UDC 539. 3

VARIATIONAL METHOD OF SOLVING A CONTACT PROBLEM FOR A COUPLED CYLINDER AND A LAYER

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It is shown that the problem of continuity of the elasticity theory equations for composite domains can be reduced to an operator equation with a positive operator in the contact stresses. By using the Ritz method, this permits reduction of the problem about the combined elastic deformation of a welded cylinder and layer to a system of linear algebraic equations. It is established that the ratio between the compressive force and the magnitude of the displacement of the cylinder upper endface, calculated by means of the approximate solution, will tend to the exact value as the order of the algebraic system increases without limit.

Problems about the contact of elastic bodies described in cylindrical coordinates have been solved in different formulations in recent years. Thus the problem of the adhering of a cylinder to a half-space has been examined in [1]. Approximate satisfaction of the continuity conditions in the contact zone is achieved by using the method of collocations, and the condition of no stresses on the cylindrical surface is reduced to an additional infinite system. The problem for an elastic half-space with an infinite cylindrical projection is reduced in [2] to a coupled infinite algebraic system and an integral equation solved by successive approximations. It should be noted that the first approximation obtained in [2] in the form of a series for the contact stresses has a logarithmic rather than a power singularity at the angular points, as it should [3,4]. Under the assumption of no tangential contact stresses, problems about the impression of an elastic cylinder in a half-space or a layer have been examined in [1,5,6].

1. General theory. Let us consider two homogeneous isotropic elastic bodies (the domain D_1, D_2) welded along a finite surface Ω (the requirements of homogeneity and isotropy are advanced only to simplify the exposition). In addition to the elasticity theory equations and the appropriate boundary conditions (generally mixed), conditions for the continuity of the displacements and stresses

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \mathbf{F}_1 = -\mathbf{F}_2 \tag{1.1}$$

should be satisfied on Ω.

The following vector-functions defined on Ω have been introduced here: $\mathbf{u}_i$ are the displacements of those points of the body D_i located on Ω, and $\mathbf{F}_i$ are the contact stresses $(i = 1, 2)$.

Let us imagine the body D_1 isolated, and let us solve the problem of determining the displacements originating for the given external effects in D_1 by assuming no stresses on the part Ω of the boundary; we denote the trace of this solution on Ω

by $\mathbf{u}_1^\circ$. We introduce the vector-function $\mathbf{u}_2^\circ$ for D_2 analogously.

Now giving the stresses $\mathbf{F}_i$ $(i = 1, 2)$ on Ω, we define the operator $\mathbf{A}_i$ for the body D_i, which compares $\mathbf{F}_i$ to the increment of the displacements of points of

$$\mathbf{A}_i\mathbf{F}_i = \mathbf{u}_i - \mathbf{u}_i^\circ \tag{1.2}$$

We consider the operator $\mathbf{A}_i$ in the Hilbert space $\mathbf{L}_2(\Omega)$, whose elements are vector-functions defined on Ω and the scalar product is given by the relationship $(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_\Omega \boldsymbol{\varphi}\cdot\boldsymbol{\psi}\, d\Omega$ (the dot is the scalar product of three-dimensional vectors); $\|\cdot\|$ is the norm in $\mathbf{L}_2(\Omega)$.

The operators $\mathbf{A}_i$ $(i = 1, 2)$ are linear, self-adjoint, and also positive, i.e., $(\mathbf{A}_i\boldsymbol{\varphi}, \boldsymbol{\varphi}) > 0$ for $\|\boldsymbol{\varphi}\| > 0$. The former is evident, while the latter follows from the Betti theorem. The third assertion is proved in an analysis of an auxiliary boundary value problem for D_i when conditions that the displacements (or stresses) be equal to zero are given on that part O_i (or B_i) of the boundary of the body D_i where geometric condition (or conditions on the stresses) underlie the problem. We then have $\mathbf{u}_i^\circ \equiv 0$, and the property of positivity of $\mathbf{A}_i$ turns out to be a corollary of the Clapeyron formula [7].

By using the operators introduced, the conjugate conditions (1.1) can be written as an operator equation with the positive operator

$$\mathbf{AF} = \mathbf{u}^\circ \tag{1.3}$$

$$(\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2,\ \mathbf{u}^\circ = \mathbf{u}_2^\circ - \mathbf{u}_1^\circ,\ \mathbf{F} = \mathbf{F}_1 = -\mathbf{F}_2)$$

Let us recall that $\mathbf{u}_i^\circ$ $(i = 1, 2)$ is determined by means of the boundary conditions of the problem, and let us also note that the second of the conditions (1.1) will be satisfied exactly, and the first approximately, for an approximate solution of (1.3).

An additional investigation shows that the operators $\mathbf{A}_i$, and therefore, $\mathbf{A}$ also, are only positive but not positive-definite (see the definition on p. 73 in [8]).

Let us show this in an example.

Example. We examine the mixed axisymmetric problem on the torsion of an elastic half-space $x < 0$ by a rigid stamp of unit radius ($\Omega = \{(\rho, \varphi, 0) : \rho \leqslant 1\}$, (ρ, φ, x) are cylindrical coordinates, and γ is the angle of stamp rotation). If it is assumed that the operator $\mathbf{A}_*\mathbf{F}_* = \mathbf{u}_*$ connecting the contact stress $\mathbf{F}_* = \{0, \tau_{x\rho}|_{x=0}, 0\}$ to the displacement $\mathbf{u}_* = \{0, \gamma\rho, 0\}$ of points of the stamp, is positive-definite, then we should have $\mathbf{F}_* \in \mathbf{L}_2(\Omega)$ [8], which is impossible since the contact stresses have a singularity which is not square integrable in this case [9].

Therefore, the problem of finding the solution of (1.3) is not correct [10]. Hence, firstly questions of the existence of a solution of this equation should be studied with reliance upon additional investigations, for instance, with the use of [11—13]; secondly, the usual (for example, projection) methods of solving (1.3) approximately may turn out to be of low efficiency for the evaluation of the contact stresses. This may elicit the necessity to use regularization methods (see [10] as well as [14], where regularizations are attained by using the extraction of stress singularities at angular points in the mixed problem of elasticity theory).

The uniqueness of the solution (1.3) follows from the positivity of the operator $\mathbf{A}$.

The above elucidation permits further application of the Ritz method to the problem of contact between a cylinder and a layer. Let us hence recall (see [8], Ch. V) that the problem of finding the solution of (1.3) can be reduced to an equivalent variational problem of seeking the minimum of the quadratic functional

$$\Psi(\mathbf{F}) = (\mathbf{AF}, \mathbf{F}) - 2(\mathbf{F}, \mathbf{u}^\circ) \tag{1.4}$$

whose approximate solution $\mathbf{F}^{(N)}$ (N is the number of coordinate functions relied upon) converges to the true solution $\mathbf{F}^*$ as $N \to \infty$ in the energy norm (in the metric of the operator $\mathbf{A}$) upon compliance with the conditions of completeness and linear independence of the system of coordinate functions, and the value of the functional Ψ, calculated by means of the approximate solution tends to its minimal value

$$\Psi(\mathbf{F}^{(N)}) = -(\mathbf{u}^\circ, \mathbf{F}^{(N)}) \downarrow \Psi_{\min}, \quad N \to \infty \tag{1.5}$$

while decreasing monotonically.

2. Formulation of the problem. Let us consider the axisymmetric problem of the joint elastic deformation of homogeneous isotropic layers of thickness H and a cylinder of radius a and height L welded to it by assuming that the upper endface of the cylinder is displaced by a quantity w°.

Referring all the linear dimensions to the cylinder radius, we introduce the system of dimensionless cylindrical coordinates (ρ, φ, x), as well as the notation

$$l = L/a, \ h = H/a, \ w_* = w^\circ/a$$

Furthermore, let us assume that there are no shear stresses at points $x = l$ of the surface, and let us pose the condition of no displacements, say, at $x = -h$. We consider the remaining parts of the elastic body surfaces stress-free.

Ascribing the superscript 1 to quantities referring to the cylinder, and the superscript 2 to the layer, we write the boundary conditions of the problem and the adjoint conditions

$$\sigma_\rho^{(1)}(1, x) = \tau_{x\rho}^{(1)}(1, x) = 0, \quad 0 \leqslant x \leqslant l \tag{2.1}$$

$$w^{(1)}(\rho, l) = w_*, \quad \tau_{x\rho}^{(1)}(\rho, l) = 0, \quad 0 \leqslant \rho \leqslant 1 \tag{2.2}$$

$$u^{(2)}(\rho, -h) = w^{(2)}(\rho, -h) = 0, \quad 0 \leqslant \rho < \infty \tag{2.3}$$

$$\sigma_x^{(2)}(\rho, 0) = \tau_{x\rho}^{(2)}(\rho, 0) = 0, \quad 1 < \rho < \infty \tag{2.4}$$

$$\sigma_x^{(1)}(\rho, 0) = \sigma_x^{(2)}(\rho, 0) = \sigma(\rho), \quad \tau_{x\rho}^{(1)}(\rho, 0) = \tau_{x\rho}^{(2)}(\rho, 0) = \tau(\rho), \quad 0 \leqslant \rho \leqslant 1 \tag{2.5}$$

$$u^{(1)}(\rho, 0) = u^{(2)}(\rho, 0), \quad w^{(1)}(\rho, 0) = w^{(2)}(\rho, 0), \quad 0 \leqslant \rho \leqslant 1 \tag{2.6}$$

Here $\{u, 0, w\}$ is the displacement vector in the cylindrical coordinate system, $\sigma_x, \tau_{x\rho}, \ldots$ are the components of the stress tensor, $\sigma(\rho)$ and $\tau(\rho)$ are unknown contact stresses.

3. Construction of the solutions. The solution in the layer can be obtained by using the Papkovich — Neuber representation and the Hankel integral (see [15], Ch.III). Omitting the sufficiently awkward calculations, let us just present the formula expressing the displacement of points of the surface $x = 0$ in terms of the contact stresses (conditions (2.3) — (2.5) are hence satisfied).

$$w^{(2)}(\rho, 0) = \frac{1}{G_2}\int_0^\infty J_0(\lambda\rho) F_1(\lambda)\, d\lambda \tag{3.1}$$

$$u^{(2)}(\rho, 0) = \frac{1}{G_2}\int_0^\infty J_1(\lambda\rho) F_2(\lambda)\, d\lambda$$

$$F_k(\lambda) = f_{k1}(\lambda h)\int_0^1 \sigma(\xi) J_0(\lambda\xi)\xi\, d\xi + f_{k2}(\lambda h)\int_0^1 \tau(\xi) J_1(\lambda\xi)\xi\, d\xi \qquad k = 1, 2$$

$$\begin{Bmatrix} f_{11}(\mu) \\ f_{22}(\mu) \end{Bmatrix} = 2(1 - \nu_2)[(3 - 4\nu_2)\operatorname{sh}\mu \operatorname{ch}\mu \mp \mu] / \Delta(\mu)$$

$$f_{12}(\mu) = f_{21}(\mu) = [(1 - 2\nu_2)(3 - 4\nu_2)\operatorname{sh}^2\mu - \mu^2]/\Delta(\mu)$$

$$\Delta(\mu) = 4(1 - \nu_2)^2 + \mu^2 + (3 - 4\nu_2)\operatorname{sh}^2\mu$$

For $\rho \in [0,1]$ the equalities (3.1) determine the specific form of the operator $\mathbf{A}_2$ introduced above.

We construct the solution for a cylinder by relying on the so-called "homogeneous solutions" satisfying the conditions (2.1) on the absence of stresses on the side surface. In constrast to the layer, the operator $\mathbf{A}_1$ for the cylinder can be constructed explicitly only in the case when the contact stresses σ, τ represent the superposition of traces of the homogeneous solutions on the surface $x = 0$

$$\sigma(\rho) = 2G_1 \sum_k C_k \sigma_k(\rho), \quad \tau(\rho) = 2G_1 \sum_k r_k C_k \tau_k(\rho) \tag{3.2}$$

In this case, by taking account of (2.2), we obtain

$$w^{(1)}(\rho, 0) - w_* = \sum_k r_k C_k w_k(\rho), \quad u^{(1)}(\rho, 0) = \sum_k C_k u_k(\rho) \tag{3.3}$$

The summation sign here extends over all integer k, C_k is a sequence of complex coefficients, C_0 is real, and all the quantities corresponding to the numbers k and $-k$ are complex conjugates. Moreover, we have introduced the notation [6]

$$\sigma_0 \equiv 1, \quad \sigma_k = -\frac{(\rho\varepsilon_k')'}{\rho}, \quad \tau_0 \equiv 0, \quad \tau_k = \varepsilon_k'$$

$$w_0 = \frac{l}{1 + \nu_1}, \quad w_k = \varepsilon_k - \delta_k, \quad u_0 = -\frac{\nu_1}{1 + \nu_1}\rho, \quad u_k = \varepsilon_k' + \delta_k'$$

$$r_0 = 1, \quad r_k = p_k \operatorname{th} p_k l, \quad k = \pm 1, \pm 2, \ldots$$

$$\varepsilon_k = \varepsilon_k(\rho) = J_0(p_k) J_0(p_k\rho) + \rho J_1(p_k) J_1(p_k\rho)$$

$$\delta_k = \delta_k(\rho) = 2(1 - \nu_1) J_1(p_k) J_0(p_k\rho) / p_k$$

where p_k are the complex roots of the equation $2[(1-\nu_1)-p^2]J_1^2(p)=p^2J_0^2(p)$ in the right half-plane, and the prime is the operator of differentiation with respect to the argument ρ.

4. Reduction of the problem to a system of linear algebraic equations. Let us use the Ritz method: assigning contact stresses in the form of finite sums of homogeneous solutions

$$\mathbf{F}^{(N)} = \{\tau^{(N)}, 0, \sigma^{(N)}\}, \quad \tau^{(N)} = 2G_1 \sum_{|k|\leqslant N} r_k C_k \tau_k(\rho) \tag{4.1}$$

$$\sigma^{(N)} = 2G_1 \sum_{|k|\leqslant N} C_k \sigma_k(\rho)$$

we calculate the value of the functional Ψ in (1.4) by means of $\mathbf{F}^{(N)}$ by taking into account that the operators $\mathbf{A}_1$, $\mathbf{A}_2$ are determined by the right sides of (3.1) and (3.3), while the conditions (2.1) – (2.4) define in the case under consideration

$$\mathbf{u}^\circ = \mathbf{u}_1^\circ = \{0, 0, w_*\}, \quad \mathbf{u}_2^\circ = 0$$

Minimizing $\Psi(\mathbf{F}^{(N)})$ in the coefficients C_k, $|k| \leqslant N$, we arrive at a complex system of linear algebraic equations which can be given the following form:

$$\sum_{|k|\leqslant N} (\gamma_{nk}^{(1)} + \gamma_{nk}^{(2)}) X_k = \begin{cases} {}^1/_2, & n=0 \\ 0, & n = \pm 1, \pm 2, \ldots, \pm N \end{cases} \tag{4.2}$$

Here

$$X_k = C_k / w_*, \quad g = G_2 / G_1$$

$$\gamma_{nk}^{(1)} = r_k \int_0^1 \sigma_n(\rho) w_k(\rho) \rho \, d\rho + r_n \int_0^1 \tau_n(\rho) u_k(\rho) \rho \, d\rho$$

$$\gamma_{nk}^{(2)} = \int_0^\infty \Phi_{nk}(\lambda) \alpha_n(\lambda) \alpha_k(\lambda) \, d\lambda, \quad \Phi_{00}(\lambda) = f_{11}(\lambda h)$$

$$\Phi_{n0}(\lambda) = \Phi_{0k}(\lambda) = f_{11}(\lambda h) - \lambda^{-1} f_{12}(\lambda h)$$

$$\Phi_{nk}(\lambda) = f_{11}(\lambda h) - 2\lambda^{-1} f_{12}(\lambda h) + \lambda^{-2} f_{22}(\lambda h)$$

$$\alpha_k(\lambda) = \int_0^1 \sigma_k(\rho) J_0(\lambda\rho) \rho \, d\rho, \quad n, k = 0, \pm 1, \pm 2, \ldots, \pm N$$

Notes. 1°. The X_0 in the system (4.2) is a real unknown and the equation corresponding to $n = 0$ contains only the real part. The determinant of the system is different from zero.

2°. The coefficient matrix of the system (4.2) is symmetric since $\gamma_{nk}^{(1)} = \gamma_{kn}^{(1)}$, $\gamma_{nk}^{(2)} = \gamma_{kn}^{(2)}$. The latter is obvious, while the former follows from the Schiff relationship of generalized orthogonality [16], which can be given the following form

$$\int_0^1 [\sigma_n(\rho) w_k(\rho) - \tau_k(\rho) u_n(\rho)] \rho d\rho = \begin{cases} 0, & n \neq k \\ 2Nn, & n = k \end{cases}$$

3°. The assumption about completeness of the system of homogeneous equations for a cylinder, expressed in [17], has been proved at this time only for the case of no displacements on the cylindrical surface [18] although analogous solutions for half-strips have been studied in more detail [19].

5. Let us turn to a calculation of the stiffness C of an elastic system by understanding C to be the ratio of the force P compressing the cylinder to the layer and the displacement w_* of the upper cylinder endface.

The approximate value of the stiffness $C^{(N)}$, calculated by means of the solution (4.1) of the system (4.2), is expressed only in terms of the zero component of this solution

$$C^{(N)} = \frac{2\pi}{w_*}\int_0^1 \sigma^{(N)}(\rho)\,\rho\,d\rho = 2\pi G_1 X_0^{(N)} \tag{5.1}$$

as is easily shown.

Moreover, let us note that the value of the functional Ψ in the approximate solution in the case under consideration is also expressed only in terms of $X_0^{(N)}$ (see (1.5))

$$\Psi(\mathbf{F}^{(N)}) = -2\pi G_1 w_*^2 X_0^{(N)} \tag{5.2}$$

Comparing (5.1) and (5.2), then on the basis of (1.5) we arrive at the following deduction which justifies the application of the Ritz method to the problem of contact between a welded cylinder and layer: the approximate value of the stiffness $C^{(N)}$, calculated by means of the solution of the system of N -th order algebraic equations), (4.2), converges to the true value of C as $N\to\infty$ by growing monotonically. The numerical determination of the quantity C can be carried out by a scheme tried out in [6].

6. Notes. Use of the Ritz method to solve the problem considered above does not permit investigation of the behavior of the stresses in the contact region. As regards the type of singularity in the stresses at the boundary of the contact zone, it is known and is the same as in the plane problem about the contact of a half- and a quarter-plane (see [20], say).

Without considering (1.3) in certain problems similar to that considered above, the solution is successfully reduced to an infinite system of linear algebraic equations upon satisfying the adjoint conditions (for example, let us mention [21, 22], where problems of the combined torsion of a cylinder and a layer are considered). A truncated infinite system hence agrees with the system obtained in solving (1.3) by the Ritz method, and the need to use existence and uniqueness criteria of the theory of infinite systems drops out (see [23], Ch. 1, Sect. 2).

In conclusion, let us note that the method of solution elucidated can be used effectively in solving different (both fundamental and mixed) problems of elasticity theory and mathematical physics in the case of domains which are a combination of subregions bounded by coordinate surfaces. Some problems of mathematical physics will reduce to equations with positive-definite operators, which will improve the quality of the approximate solutions obtained.

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